CCNY-HEP-94/10
October 1994

CHERN-SIMONS AND WZNW THEORIES AND THE QUARK-GLUON PLASMA *

V.P. NAIR
Physics Department
City College of the City University of New York
New York, New York 10031, U.S.A.

ABSTRACT

Summation over hard thermal loops, by themselves and as insertions in higher order Feynman diagrams, is important in thermal perturbation theory for Quantum Chromodynamics, so that all contributions of a given order in the coupling constant can be consistently taken into account. I review some of the basic properties of hard thermal loops and how the generating functional for them is related to the eikonal for a Chern-Simons gauge theory, and using an auxiliary field, to the gauged Wess-Zumino-Novikov-Witten action. The Hamiltonian analysis of the effective action and a discussion of plasma waves are given. It is also pointed out that a possible expression for the magnetic mass term can be written in a closely related way.

Contents

* Lectures presented at Mt. Sorak Summer School, S. Korea, June 27-July 2, 1994

1. Introduction

In these lectures, I review the relationship between Chern-Simons (CS) and Wess-Zumino-Novikov-Witten (WZNW) theories and Quantum Chromodynamics (QCD) at finite temperature [1−3]. The results are expected to be useful in understanding the theory of the quark-gluon plasma.

There is essentially universal agreement by now that QCD is the theory of strong interactions. One of the interesting features of QCD is the deconfinement phase transition which is indicated by general theoretical arguments as well as lattice simulations [4,5]. If the ambient temperature is raised beyond a critical value T_C, hadronic matter makes a phase transition. Quarks and gluons are no longer confined; we have a plasma of quarks, antiquarks and gluons. As a new phase of hadronic matter, the quark-gluon plasma is of considerable interest to physicists.

The plasma phase of hadronic matter, it is generally believed, can be achieved in the collisions of sufficiently heavy nuclei at sufficiently high energy. However, the key word here is *sufficiently*. Will the relativistic heavy ion collider (RHIC), to be built at Brookhaven, have enough energy to achieve this? Can thermal equilibrium be obtained even for a short duration before the plasma cools and rehadronizes? These are questions to which, as of now, there are no definitive and clear answers [6]. Nevertheless, understanding physical phenomena in the quark-gluon plasma at and near thermal equilibrium, which can be described by QCD at high temperatures, is an important and necessary first step. The critical temperature T_C for the transition to the plasma phase is expected to be of the order of the scale parameter, Λ_{QCD} which is around 200 MeV. Physical phenomena near this temperature are of great interest, especially since they may serve as signatures for the transition [7]. We shall not discuss these however. Instead, we shall consider temperatures which are significantly higher than T_C; the average momentum carried by quarks and gluons will be much higher than Λ_{QCD} and one can expect thermal perturbation theory to provide an adequate description of the physics. We are interested in the technical problems of thermal perturbation theory for QCD, especially hard thermal loops and related physical phenomena such as Debye screening, Landau damping, propagation of plasma waves, etc. (Eventhough we consider $T \gg T_C$, many of the results are of importance near T_C as well. It is qualitatively easier to isolate the relevant terms at high temperatures; the modifications required at lower temperatures can be easily included. The quark flavors of interest are also the light quarks, with masses small compared to T. The heavier flavors have the standard Boltzmann suppression factors and lead to small modifications.)

As is well known , among the most elementary phenomena in a plasma are Debye screening and Landau damping. If we consider an Abelian plasma of positive and negative charges $\pm e$, viz., electrodynamics, screening can be understood using the classic argument of Debye. One considers the Poisson equation for the electrostatic potential of a test charge, say positive, in the plasma.

$$-\nabla^2 \; A_0 \;=\; ne \left(\frac{e^{eA_0/T} \;-\; e^{-eA_0/T}}{e^{eA_0/T} \;+\; e^{-eA_0/T}} \right) \tag{1a}$$

$$\approx \left(\frac{ne^2}{T}\right) A_0 \tag{1b}$$

where the right hand side is the charge density in the vicinity of the test charge. n is the average number density of particles. The exponentials are the Boltzmann factors giving the preferential accumulation of negative and depletion of positive charges in the vicinity due to the Coulomb forces, with proper normalization. The approximation (1b), which is valid for high temperatures, shows that the solutions have the screened Coulomb form $(1/r)\exp(-m_D r)$ with a Debye screening mass $m_D^2 = (ne^2/T)$. For a relativistic plasma, the qualitative features of this argument are valid and with $n \sim T^3$, we expect $m_D^2 \sim e^2T^2$. And by calculating the photon propagator in thermal electrodynamics, one can indeed obtain a similar result [8].

The above argument is simple and nice, but is presented in terms of potentials and as such, it does not seem to be gauge invariant. For the Abelian case, one can easily reformulate the arguments using only the gauge-invariant electric and magnetic fields. However, in a non-Abelian plasma, such as the quark-gluon plasma, it is difficult to avoid the use of gauge potentials altogether. Further, since even the notion of the charge of a gluon has to be defined with respect to some chosen Abelian direction of the gauge group, it is clear that the simple argument of Eq.(1) will have to be modified. The question of interest to us is: how do we obtain a gauge-invariant description of Debye screening in a non-Abelian plasma (in terms of gauge potentials)? More specifically, we need a functional of the gauge potentials, $\Gamma[A]$, which is generated by the statistical distributions and is effectively a gauge-invariant mass term for the gauge fields. Screening, of course, is the static part of a more general problem, the dynamical part of which is the propagation of plasma waves. Such a $\Gamma[A]$ will thus be important for the discussion of plasma waves as well.

Landau damping arises from the scattering of the particles in the plasma against an external field. This can transfer energy and momentum from the field to the particle, leading to damping of the field. The process is essentially the inverse of Čerenkov radiation and can be described by the proper continuation of the results for plasma waves to a region of spacelike momenta. Of course, being a time-asymmetric process, one must describe it in terms of the equations of motion rather than an action; the relevant current for these equations will be a continuation of what is given by $\Gamma[A]$.

Hard thermal loops are closely related to the above. They arise because of the need to carry out a partial resummation of perturbation theory in thermal QCD. The need for resummation is easily seen using the following argument due to Pisarski [9]. Consider the elementary polarization diagram of gluons, fig.(1a) and its first correction fig.(1b). Before the final loop integration over p, the ratio of diagram (1b) to (1a) is $\Pi(p)/p^2$. Because of Debye screening, the polarization tensor $\Pi(p)$ has a small p-expansion, $\Pi(p) \sim g^2T^2 \; + \; g^2T|p| \; + ...$ (g is the quark-gluon or gluon-gluon coupling constant.) We thus see that for the small p-regime of integration, i.e., $p^2 \leq g^2T^2$, the naively higher order diagram (1b) is comparable to the lower order term (1a). Therefore, to be consistent to a given order in g, one must sum over

(1a), (1b), and a series of further insertions of $\Pi(p)$. This resummation leads to new effective propagators; in general, we shall need new effective vertices as well. The resummation can be summarized by saying that the new propagators and vertices arise from an action

$$S \; = \; \int d^4x \; \left(-\tfrac{1}{4}F^2\right) \; + \; \Gamma[A] \tag{2}$$

where we add a term $\Gamma[A]$ to the standard Yang-Mills action for gluons.

Now, going back to figures (1), we see that the regime of interest is when the momentum external to $\Pi(p)$, viz., p is of the order gT or less. We can thus take the momenta carried by the potentials in $\Gamma[A]$ to be $\lesssim gT$. In order to determine which diagrams can contribute to $\Gamma[A]$, we shall need to extend the argument given above for $\Pi(p)$ to general vertices and obtain rules for counting powers of g and T, keeping in mind the restriction on external momenta. These rules basically follow from the thermal propagator and let us briefly examine them.

2. Analysis of diagrams and power counting rules

The propagator for a quark at finite temperature is defined by

$$S(x,y) = \langle \mathcal{T} \; q(x)\bar{q}(y)\rangle \tag{3}$$

where $\mathcal{T}$ denotes time-ordering. We shall use Minkowski space propagators with thermal averages for the products of creation and annihilation operators; this is conceptually the simplest for us since we need to talk about nonequilibrium phenomena such as the propagation of plasma waves. The propagator can be evaluated as

$$\begin{aligned} S(x,y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\{&\Theta(x^0-y^0)[\alpha_p e^{-ip\cdot(x-y)}\gamma\cdot p + \bar{\beta}_p e^{ip'\cdot(x-y)}\gamma\cdot p'] - \\ &\Theta(y^0-x^0)[\beta_p e^{-ip\cdot(x-y)}\gamma\cdot p + \bar{\alpha}_p e^{ip'\cdot(x-y)}\gamma\cdot p']\} \end{aligned} \tag{4}$$

where $p^0 = |\vec{p}|,\ p = (p^0,\vec{p}),\ p' = (p^0,-\vec{p})$, and $\Theta(x)$, of course, is the step function. Also

$$\alpha_p = 1 - n_p, \qquad \beta_p = n_p. \tag{5}$$

The distribution functions $n_p, \bar{n}_p$ are defined by the thermal averages

$$\begin{aligned} \langle a_p^{\dagger\alpha,r} a_p^{\beta,s}\rangle &= n_p \delta^{rs}\delta^{\alpha\beta} \\ \langle b_p^{\dagger\alpha,r} b_p^{\beta,s}\rangle &= \bar{n}_p \delta^{rs}\delta^{\alpha\beta} \end{aligned} \tag{6}$$

where $(a_p^{\dagger\alpha,r}, a_p^{\alpha,r}), (b_p^{\dagger\alpha,r}, b_p^{\alpha,r})$ are the annihilation and creation operators for quarks and antiquarks respectively. α,β are spin indices; r,s are color indices. For a plasma of zero fermion number, we can take

$$n_p = \bar{n}_p = \frac{1}{e^{p^0/T}+1} \tag{7}$$

For a plasma with a nonzero value of fermion number, there is a chemical potential and correspondingly $n_p, \bar{n}_p$ are not equal.

We can also write the propagator, for zero chemical potential, as

$$S(x,y) = \int \frac{d^4p}{(2\pi)^4}\; e^{-ip(x-y)} \left[i\frac{\gamma \cdot p}{p^2 + i\epsilon} \; - \; \gamma \cdot p \; 2\pi \; n_p \delta(p^2) \right] \tag{8}$$

The first term in brackets is the standard Feynman term at $T = 0$. The T-dependent correction is effectively on mass-shell, enforced by the δ-function.

For the gluon field, the propagator in the Feynman gauge can be similarly written as $G^{ab}_{\mu\nu}(x,y) = \delta^{ab} g_{\mu\nu} G(x,y)$ where $g_{\mu\nu}$ is the metric tensor, a, b are color indices and

$$\begin{aligned} G(x,y) \;&=\; \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \Bigg[[\Theta(x^0 - y^0) \; \{\alpha_p e^{-ip(x-y)} + \beta_p e^{ip(x-y)}\} \\ &\qquad + \Theta(y^0 - x^0) \; \{\beta_p e^{-ip(x-y)} + \alpha_p e^{ip(x-y)}\} \Bigg] \qquad (9a) \\ &= \int \frac{d^4p}{(2\pi)^4} \left[\frac{i}{p^2 + i\epsilon} \; + \; 2\pi \; n_p \; \delta(p^2) \right] \qquad (9b) \end{aligned}$$

Here $\alpha_p = 1 + n_p,\ \beta_p = n_p$ and n_p is the bosonic distribution function

$$n_p = \frac{1}{e^{p^0/T} - 1} \tag{10}$$

For our power counting rules, it is easiest to consider the propagators with the time-orderings separated as in Eqs.(4,9a). Suppose now that these propagators are in a loop diagram, with integration over momenta. The distributions tell us that the average momentum is of the order of T. For each diagram, we thus get [9]:

1) T^3 corresponding to d^3p

2) $1/T$ for each propagator from the $1/2p^0$ factor

3) a power of T for each factor of p_μ in the numerators in vertices

4) for each propagator, other than the first, a power of $1/k$, where k is the external momentum ($\sim gT$)

5) a factor k/T, one for each loop, for two or more propagators of the same statistical type.

The last two rules can be understood as follows. In using Eqs.(4,9a) for the propagators in any Feynman diagram, we encounter many terms corresponding to different ordering of the time arguments $x^0, y^0, z^0, etc.$ An obvious strategy for simplification is to carry out the time-integrations first, introducing convergence factors $e^{\pm\epsilon x^0}, e^{\pm\epsilon y^0}$ etc., ϵ small and positive, as required. The integrations give energy-denominators and bring the result to a form where simplification due to the external momenta being soft ($\sim gT$) relative to the loop momenta ($\sim T$) can be implemented easily. Among energy-denominators, there will be some of the form $p^0 - q^0 \pm k^0$, where p, q refer to loop momenta. Because their difference is involved, such a denominator is of the order of k, eventhough $p^0,\ q^0$ can be of the order

of T. This gives the $1/k$-factor of rule 4. If we have only one propagator, time-integration gives only an energy-conservation δ-function; thus the $1/k$-factors are only for propagators other than the first. The last rule has to do with the fact that the difference of statistical distributions such as $n_p - n_q$ is what arises. With $p - q \approx k$, this gives rule 5. The last two rules will become clearer in the course of the evaluation of the two-point function which we shall do shortly.

We can apply these rules, as an example, to the three-point function and its first correction as shown. Because of the derivative coupling, fig.(2a) goes as $gk \sim g^2 T$. For fig.(2b), first of all we have g^3, then T^3 from rule 1, $1/T^3$ from rule 2, T^3 from rule 3, $1/k^2$ from rule 4 and k/T from rule 5. With $k \sim gT$, this is again $\sim g^2 T$. Thus fig.(2b) must be included in our calculation of $\Gamma[A]$. Continuing in a similar way, one can see that all diagrams which should be included in Γ are one-loop diagrams [9]; further, only the T-dependent terms of these one-loop diagrams (and of course with external momenta small compared to T) are important. Thermal one-loop diagrams with loop momenta $\sim T$ and external momenta $\lesssim gT$ are called hard thermal loops.

3. Calculation of the two-point function

Let us now carry out an explicit calculation, say for the two-point function. This may be a trifle tedious, but is well worth the effort since it illustrates many of the features which generalize to n-point functions [1]. We consider a one-loop quark graph with two external gluon lines, in other words, the gluon polarization diagram. The relevant part of the Lagrangian for the quark fields $q, \bar{q}$ is

$$\mathcal{L} = \bar{q}\ i\gamma \cdot (\partial + A)\ q \tag{11}$$

where $A_\mu = -it^a A^a_\mu$ is the Lie-algebra-valued gluon vector potential, t^a are hermitian matrices corresponding to the generators of the Lie algebra in the representation to which the quarks belong; $[t^a, t^b] = if^{abc}t^c$, where f^{abc} are the structure constants, and $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$. We take the gauge group to be $SU(N)$ with N_F flavors of quarks. From now on, we shall not explicitly display the quark-gluon coupling constant g, as it is easily recovered at any stage by $A \to gA$. The one-loop quark graphs are given by the effective action

$$\Gamma = -i\ \mathrm{Tr}\ \log\ (1 + S\ \gamma \cdot A) \tag{12}$$

The two-gluon term in Γ is given by

$$\Gamma^{(2)} = \frac{i}{2} \int d^4x d^4y\ \mathrm{Tr}\,[\ \gamma \cdot A(x) S(x,y) \gamma \cdot A(y) S(y,x)] \tag{13}$$

In using Eq.(4) for the propagator, we find four terms in $\Gamma^{(2)}$ with $x^0 > y^0$ and four terms with $y^0 > x^0$. Writing

$$A_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\ e^{ikx}\ A_\mu(k) \tag{14}$$

and carrying out the time-integrations we get

$$\Gamma^{(2)} = -\tfrac{1}{2}\int d\mu(k)\int \frac{d^3q}{(2\pi)^3}\frac{1}{2p^0}\frac{1}{2q^0}\Bigg[\ T(p,q)\Big(\frac{\alpha_p\beta_q}{p^0-q^0-k^0-i\epsilon}-\frac{\alpha_q\beta_p}{p^0-q^0-k^0+i\epsilon}\Big)+ \\ T(p,q')\Big(\frac{\alpha_p\bar\alpha_q}{p^0+q^0-k^0-i\epsilon}-\frac{\beta_p\bar\beta_q}{p^0+q^0-k^0+i\epsilon}\Big)+ \\ T(p',q)\Big(\frac{\bar\alpha_p\alpha_q}{p^0+q^0+k^0-i\epsilon}-\frac{\bar\beta_p\beta_q}{p^0+q^0+k^0+i\epsilon}\Big)+ \\ T(p',q')\Big(\frac{\bar\alpha_p\bar\beta_q}{p^0-q^0+k^0-i\epsilon}-\frac{\bar\beta_p\bar\alpha_q}{p^0-q^0+k^0+i\epsilon}\Big)\Bigg] \tag{15}$$

where

$$T(p,q) = \mathrm{Tr}[\ \gamma\cdot A(k)\gamma\cdot p\gamma\cdot A(k')\gamma\cdot q\ ] \tag{16}$$

$$d\mu(k) = (2\pi)^4\delta^{(4)}(k+k')\frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4} \tag{17}$$

In Eq.(15), $\vec p = \vec q + \vec k$. Since $p^0 = |\vec q + \vec k| \simeq q^0 + \vec q\cdot\vec k/q^0$ for $|\vec k|$ small compared to $|\vec q|$, the denominators in Eq.(15) involve $k\cdot Q$, $k\cdot Q'$ and $2q^0 + k\cdot Q$, $2q^0 + k\cdot Q'$ where

$$Q = (1,\ \frac{\vec q}{q^0}),\ Q' = (1,\ -\frac{\vec q}{q^0}) \tag{18}$$

Notice that Q^μ, Q'^μ are null vectors, $Q^\mu Q_\mu = Q'^\mu Q'_\mu = 0$. This is a remnant of the δ-function in Eq.(8) enforcing the mass-shell condition for the quarks.

The $i\epsilon$'s in the denominators in Eq.(15) will be let go to zero at this stage. The $i\epsilon$'s can contribute to the imaginary part of the two-point function, corresponding to the Landau damping of the gluon field. The retarded rather than time-ordered propagators are appropriate for a discussion of damping effects; we shall obtain the correct imaginary part by an appropriate continuation later. For the moment, let us ignore the $i\epsilon$'s. Using α,β from Eq.(5), we find, for the temperature-dependent part of $\Gamma^{(2)}$,

$$\Gamma^{(2)} = -\tfrac{1}{2}\int d\mu(k)\int\frac{d^3q}{(2\pi)^3}\frac{1}{2p^0}\frac{1}{2q^0}\ \Bigg[(n_q-n_p)\frac{T(p,q)}{p^0-q^0-k^0}+(\bar n_q-\bar n_p)\frac{T(p',q')}{p^0-q^0+k^0}- \\ (n_p+\bar n_q)\frac{T(p,q')}{p^0+q^0-k^0}-(\bar n_p+n_q)\frac{T(p',q)}{p^0+q^0+k^0}\Bigg]. \tag{19}$$

We see that the result is linear in the distribution functions, a property which holds in general for the n-point functions. Notice also how soft denominators like $p^0 - q^0 \pm k^0$ arise from the time-integration. For $|\vec k|$ small compared to the loop momentum $|\vec q|$, we can write

$$p^0-q^0-k^0 \simeq -k\cdot Q \qquad p^0-q^0+k^0\simeq k\cdot Q'$$

$$p^0+q^0\pm k^0 \simeq 2q^0 \tag{20}$$

$$
\begin{aligned}
T(p,q) &\simeq 8{q^0}^2\,\mathrm{tr}(A_1\cdot Q A_2\cdot Q)\\
T(p',q') &\simeq 8{q^0}^2\,\mathrm{tr}(A_1\cdot Q' A_2\cdot Q') \qquad (21)\\
T(p',q)\simeq T(p,q') &\simeq 4{q^0}^2\,\mathrm{tr}(A_1\cdot Q' A_2\cdot Q + A_1\cdot Q A_2\cdot Q' - 2A_1\cdot A_2)
\end{aligned}
$$

where $A_1 = A(k)$, $A_2 = A(k')$ and the remaining trace in the expressions for T's, denoted by tr, is over color indices. The difference of distribution functions can also be approximated as $n_p - n_q \simeq \frac{dn}{dq^0}\vec{Q}\cdot\vec{k}$. (This is the extra factor of k/T mentioned as rule 5 above.) Using these results, Eq.(19) simplifies to

$$
\Gamma^{(2)} = -\tfrac{1}{2}\int d\mu(k)\int \frac{d^3q}{(2\pi)^3}\ \mathrm{tr}\Big[\ \Big(\frac{dn}{dq^0}\frac{A_1\cdot QA_2\cdot Q}{k\cdot Q} - \frac{d\bar{n}}{dq^0}\frac{A_1\cdot Q'A_2\cdot Q'}{k\cdot Q'}\Big)2\vec{Q}\cdot\vec{k}
- \frac{n+\bar{n}}{q^0}(A_1\cdot Q'A_2\cdot Q + A_1\cdot QA_2\cdot Q' - 2A_1\cdot A_2)\Big] \qquad (22)
$$

We have the result

$$
\int d^3q \frac{dn}{dq^0} f(Q) = -\int d^3q \frac{2n}{q^0} f(Q) \qquad (23)
$$

for any function f of Q, or Q'. We can further use $2\vec{Q}\cdot\vec{k} = k\cdot Q' - k\cdot Q$. Eq.(22) then simplifies to

$$
\Gamma^{(2)} = -\tfrac{1}{2}\int d\mu(k)\int \frac{d^3q}{(2\pi)^3}\ \mathrm{tr}\Big[\ \frac{n+\bar{n}}{q^0}(2A_1\cdot QA_2\cdot Q - A_1\cdot QA_2\cdot Q' - A_1\cdot Q'A_2\cdot Q + 2A_1\cdot A_2)
- \frac{n}{q^0}2A_1\cdot QA_2\cdot Q\frac{k\cdot Q'}{k\cdot Q} - \frac{\bar{n}}{q^0}2A_1\cdot Q'A_2\cdot Q'\frac{k\cdot Q}{k\cdot Q'}\Big]. \qquad (24)
$$

The angular integration in Eq.(24) over the directions of $\vec{q}$ (or $\vec{Q}$) can help in simplifying it further by virtue of

$$
\int d\Omega\ (2A_1\cdot QA_2\cdot Q - A_1\cdot QA_2\cdot Q' - A_1\cdot Q'A_2\cdot Q + 2A_1A_2) = \int d\Omega\ (2A_1\cdot QA_2\cdot Q') \qquad (25)
$$

Defining

$$
A_+ = \frac{A{\cdot}Q}{2}, \qquad A_- = \frac{A{\cdot}Q'}{2} \qquad (26)
$$

we can write Eq.(24) as

$$
\Gamma^{(2)} = -\tfrac{1}{2}\int d\mu(k)\int \frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\ 16\ \mathrm{tr}\Big[\ A_{1+}A_{2-}(n+\bar{n}) - n\frac{k\cdot Q'}{k\cdot Q}A_{1+}A_{2+} - \bar{n}\frac{k\cdot Q}{k\cdot Q'}A_{1-}A_{2-}\Big]. \qquad (27)
$$

This expression becomes more transparent when written in coordinate space and finally with a Wick rotation to Euclidean space. Let us define the Green's functions

$$
\begin{aligned}
G(x_1,x_2) &= \int \frac{e^{-ip\cdot(x_1-x_2)}}{p\cdot Q}\frac{d^4p}{(2\pi)^4} \\
G'(x_1,x_2) &= \int \frac{e^{-ip\cdot(x_1-x_2)}}{p\cdot Q'}\frac{d^4p}{(2\pi)^4}
\end{aligned} \tag{28}
$$

In terms of the null vectors $Q=(1,\vec{q}/q^0),\ Q'=(1,-\vec{q}/q^0)$, we can introduce the lightcone coordinates (u,v,x^T) as

$$
u=\frac{Q'\cdot x}{2}, \qquad v=\frac{Q\cdot x}{2}, \qquad \vec{Q}\cdot\vec{x}^T=0 \tag{29}
$$

where $\vec{Q}=\vec{q}/q^0$. We then have $Q\cdot\partial=\partial_u,\ Q'\cdot\partial=\partial_v$. We shall also introduce a Euclidean version of our results by the correspondence

$$
\begin{aligned}
2u &\leftrightarrow z, \qquad 2v \leftrightarrow \bar{z} \\
\partial_u = Q\cdot\partial &\leftrightarrow 2\partial_z, \qquad \partial_v = Q'\cdot\partial \leftrightarrow 2\partial_{\bar{z}}
\end{aligned} \tag{30}
$$

The Green's functions in Eq.(28) are the continuations of the Euclidean functions

$$
\begin{aligned}
G_E(x_1,x_2) &= \frac{1}{2\pi i}\frac{\delta^{(2)}(x_1^T-x_2^T)}{(\bar{z}_1-\bar{z}_2)} \\
G'_E(x_1,x_2) &= \frac{1}{2\pi i}\frac{\delta^{(2)}(x_1^T-x_2^T)}{(z_1-z_2)}
\end{aligned} \tag{31}
$$

More precisely, the Green's functions in Eq.(28) obey the equations $Q\cdot\partial G(x_1,x_2) = -i\delta(x_1-x_2),\ Q'\cdot\partial G'(x_1,x_2) = -i\delta(x_1-x_2)$. The Green's functions in Eq.(31) are the solutions to the corresponding Euclidean equations $2\partial_z G_E(x_1-x_2) = -i\delta(x_1-x_2),\ 2\partial_{\bar{z}} G'_E(x_1-x_2) = -i\delta(x_1-x_2)$. This leads to the correspondence

$$
\begin{aligned}
\frac{k\cdot Q'}{k\cdot Q} &\longleftrightarrow \frac{1}{\pi}\ \frac{1}{\bar{z}_{12}\bar{z}_{21}} \\
\frac{k\cdot Q}{k\cdot Q'} &\longleftrightarrow \frac{1}{\pi}\ \frac{1}{z_{12}z_{21}} \\
\frac{1}{k\cdot Q} &\longleftrightarrow \frac{1}{2\pi i}\ \frac{1}{\bar{z}_{12}}
\end{aligned} \tag{32}
$$

where $z_{ij}=z_i-z_j$, etc. Eq.(27) for $\Gamma^{(2)}$ can then be written as

$$
\begin{aligned}
\Gamma^{(2)} = -\tfrac{1}{2}\int \frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\ 16\ \mathrm{tr}\Bigg[\ &(n+\bar{n})\ \int d^4x\ A_+(x)A_-(x) \\
-n\ \pi\int d^2x^T\frac{d^2z_1}{\pi}\frac{d^2z_2}{\pi}\frac{A_+(x_1)A_+(x_2)}{\bar{z}_{12}\bar{z}_{21}} &- \bar{n}\ \pi\int d^2x^T\frac{d^2z_1}{\pi}\frac{d^2z_2}{\pi}\frac{A_-(x_1)A_-(x_2)}{z_{12}z_{21}}\Bigg].
\end{aligned} \tag{33}
$$

This is finally starting to look nice. Let us define the functional $I(A_+)$ by the formula

$$
\begin{aligned}
I(A_+) &= i\sum \frac{(-1)^n}{n}\int\ d^2x^T\ \frac{d^2z_1}{\pi}\cdots\frac{d^2z_n}{\pi}\ \frac{\mathrm{tr}(A_+(x_1)\cdots A_+(x_n))}{\bar z_{12}\bar z_{23}\cdots\bar z_{n1}}\\
&= \frac{i}{2}\int\ d^2x^T\ \frac{d^2z_1}{\pi}\frac{d^2z_2}{\pi}\ \frac{\mathrm{tr}(A_+(x_1)A_+(x_2))}{\bar z_{12}\bar z_{21}}+\cdots
\end{aligned}
\tag{34}
$$

We shall show later that $I(A_+)$ is related to the eikonal for a Chern-Simons theory. Eq.(33) for $\Gamma^{(2)}$ can finally be written as

$$
\Gamma^{(2)} = \int \frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\quad K^{(2)}[A_+,A_-] \tag{35a}
$$

where

$$
K[A_+,A_-] = -16\ \left[\ \frac{(n+\bar n)}{2}\int d^4x\ \mathrm{tr}\ \big(A_+(x)A_-(x)\big) + n\ i\pi I(A_+) + \bar n\ i\pi \tilde I(A_-)\right] \tag{35b}
$$

$K^{(2)}[A_+,A_-]$ in Eq.(35a) denotes terms in K which are quadratic in A. $\tilde I$ is obtained from I by $z\leftrightarrow\bar z$.

Although we have introduced different distribution functions $n,\bar n$ for quarks and antiquarks, it is only $n+\bar n$ which is relevant at high temperatures. We see that, by virtue of $\int d\Omega\ I(A_+) = \int d\Omega\ \tilde I(A_-)$, we can write Eq.(35b) as

$$
K[A_+,A_-] = -16\ \frac{n+\bar n}{2}\ \left[\ \int d^4x\ \mathrm{tr}\big(A_+(x)A_-(x)\big) + i\pi I(A_+) + i\pi\tilde I(A_-)\right]. \tag{36}
$$

The integral over the magnitude of $\vec q$ in Eqs.(35) can be easily carried out. With zero chemical potential,

$$
\Gamma^{(2)} = \frac{-T^2}{12\pi}\int d\Omega\ \left[\ \int d^4x\ \mathrm{tr}\big(A_+\ A_-\big) + i\pi I^{(2)}(A_+) + i\pi\tilde I^{(2)}(A_-)\right]. \tag{37}
$$

4. Properties of $\Gamma[A]$

We can extract certain properties of $\Gamma[A]$ from the power counting rules discussed earlier. There are two key properties which are important to our analysis.

1) $\Gamma[A]$ is gauge-invariant with respect to gauge transformations of the gauge potential A_μ and is independent of the gauge-fixing used to define the gluon propagators [9,10].

We can understand how the kinematics of hard thermal loops can lead to gauge invariance. The thermal propagators satisfy the same differential equations as the zero temperature propagators. This is evident from Eqs.(8,9b); only the choice of the homogeneous solution, equivalently boundary condition, is different. As a result, the generating functional of one-particle irreducible vertices, viz., $\Gamma[A,c,\bar c,q,\bar q]$ obeys the standard BRST Ward identities. c and $\bar c$ are the ghost and antighost fields respectively. For the usual gauge fixing term $\lambda(\partial\cdot A)^2$, we thus have [11]

$$
\int d^4x\ \left[\frac{\delta\Gamma}{\delta A_\mu}\frac{\delta\Gamma}{\delta K_\mu}\ +\ \frac{\delta\Gamma}{\delta c}\frac{\delta\Gamma}{\delta L}\ -\lambda(\partial\cdot A)\frac{\delta\Gamma}{\delta\bar c}\right]\ = 0 \tag{38}
$$

K_μ and L are sources, corresponding to the terms $K^a_\mu (D_\mu c)^a$, $f^{abc}L^a c^b c^c$ added to the action. In the hard thermal loop approximation, terms involving the ghosts are subdominant. Recall that the ghost-gluon coupling involves $f^{abc}A^{a\mu}(\partial_\mu \bar{c}^b)c^c$; the derivative is on the antighost field. In a diagram with external ghosts, this is a power of external momentum and therefore such a diagram is smaller compared to a similar diagram with the ghost lines replaced by gluons where there are contributions with all derivatives on the internal lines which give powers of T. (See figs.(3).) The terms $\frac{\delta\Gamma}{\delta c}$, $\frac{\delta\Gamma}{\delta \bar{c}}$ in Eq.(38) are thus negligible in the hard thermal loop approximation. Further, we are interested only in one-loop terms. The identity (38) then gives the gauge invariance of Γ. Thus effectively, in a high-T-expansion, the leading term $\Gamma[A]$ which is proportional to T^2 is gauge-invariant. Notice also that since the thermal contribution to the propagator is on-shell, the T-dependent part of a one-loop diagram is classical and so it is not surprising that the BRST Ward identities reduce to the statement of gauge invariance. The fact that Γ does not depend on the gauge choice for the gluon propagators can be seen by similar arguments; identities for the variation of Γ for changes in λ can be written down and simplified [9,10].

2) $\Gamma[A]$ has the form [12]

$$\Gamma = (N + \tfrac{1}{2}N_F)\ \frac{T^2}{12\pi}\ \left[\int d^4x\ 2\pi A^a_0 A^a_0 + \int d\Omega\ W(A\cdot Q)\right]. \tag{39}$$

Here Q_μ is the null vector $(1,\vec{Q})$ with $\vec{Q}^2 = 1,\ Q_\mu Q_\mu = 0$. The $A^a_0 A^a_0$-term is the lowest order Debye screening effect; it is clearly an electrostatic mass term. The key point about Eq.(39) is that, for each $\vec{Q}$, $\Gamma[A]$ involves essentially only two components of the gauge potential, A_0 and $A\cdot Q$. The $d\Omega$-integration will bring in all components of the potential, but for each $\vec{Q}$ only two components are needed. We have already seen this structure in the explicit calculation of the two-point function, where $\vec{Q}$ was the angular part of the loop-momentum $\vec{q}$. The $d\Omega$-integration in Eq.(39) is over the orientations of $\vec{Q}$ and is the unfinished part of the loop integration, after the integration over the modulus of the loop momentum has been done. The structure of Eq.(39) can be seen by analysis of diagrams again. For example, for the diagram of fig.(3b), with the derivative gluon coupling, since $p\cdot A \approx q\cdot A$, the possible tensor structures are q^2A^2 and $(q\cdot A)^2$. The former is zero since the propagator involves the δ-function $\delta(q^2)$. Writing $q = |\vec{q}|(1,\vec{Q})$ and carrying out the $|\vec{q}|$-integration, we are left with a structure like Eq.(39). This argument generalizes to diagrams with arbitrary number of external gluons.

Given these two properties of Γ one can determine W and hence Γ simply by the requirement of gauge invariance [12]. The condition for gauge invariance of Γ is

$$\int d\Omega\ \delta W = 4\pi \int d^4x\ \dot{A}^a_0 \omega^a \tag{40}$$

where $\dot{A}^a_0$ is the time-derivative of A^a_0 and $\omega = -it^a\omega^a$ is the parameter of the gauge transformation, i.e., $\delta A_\mu = \partial_\mu \omega + [A_\mu, \omega]$. Eq.(40) can be realized by

$$\delta W = \int d^4x\ \dot{A}^a \cdot Q\omega^a. \tag{41}$$

One can check that Eq.(41) is indeed the way gauge invariance is realized by analysis of the diagrams. It is clearly so for the two-point function from our explicit calculation. We now rewrite Eq.(41), using

$$\delta W = -\int d^4x (Q \cdot \partial \frac{\delta W}{\delta(A \cdot Q)} + [A \cdot Q, \frac{\delta W}{\delta(A \cdot Q)}])^a \omega^a \tag{42}$$

as

$$\frac{\partial f}{\partial u} + [A \cdot Q, f] + \frac{1}{2}\frac{\partial(A \cdot Q)}{\partial v} = 0, \tag{43}$$

where

$$f = \frac{\delta W}{\delta(A \cdot Q)} + \tfrac{1}{2} A \cdot Q. \tag{44}$$

We have used the lightcone coordinates from Eq.(29). We now make the Wick rotation as in Eq.(30) which gives $A_+ = \frac{1}{2} A \cdot Q \ \rightarrow A_z$; also rename f as

$$a_{\bar{z}} = -f = -\frac{1}{2}\frac{\delta W}{\delta A_z} - A_z \tag{45}$$

The condition of gauge invariance, Eq.(43), then becomes

$$\partial_{\bar{z}} A_z - \partial_z a_{\bar{z}} + [a_{\bar{z}}, A_z] = 0. \tag{46}$$

If A_z, $a_{\bar{z}}$ are thought of as the gauge potentials of another gauge theory, we see that Eq.(46) is the vanishing of the field strength or curvature $F_{z\bar{z}}$. The gauge theory whose equations of motion say that the field strengths vanish is the Chern-Simons theory. We shall therefore put aside Eq.(46) for a moment and turn to a short digression on the Chern-Simons theory, returning to Eq.(46) and its solution in section 6. Of course, an understanding of Chern-Simons theory is not absolutely essential to solving Eq.(46). One can simply solve Eq.(46) and regard Chern-Simons theory as an interpretation of the mathematical steps along the way. However Chern-Simons theory does illuminate many of the nice geometrical properties of the final result and is a worthwhile digression.

5. Chern-Simons and WZNW Theories

The Chern-Simons theory is a gauge theory in two space (and one time) dimensions [13,14]. The action is given by

$$S = \frac{\kappa}{4\pi} \int_{M \times [t_i, t_f]} d^3x \ \epsilon^{\mu\nu\alpha} \ \mathrm{tr}(a_\mu \partial_\nu a_\alpha + \tfrac{2}{3} a_\mu a_\nu a_\alpha). \tag{47}$$

Here a_μ is the Lie algebra valued gauge potential, $a_\mu = -it^a a^a_\mu$. κ is a constant whose precise value we do not need to specify at this stage. We shall consider the spatial manifold to be $\mathbf{R}^2$, or $\mathbf{C}$ since we shall be using complex coordinates $z = x + iy$, $\bar{z} = x - iy$. (Actually, we have sufficient regularity conditions at spatial infinity that

we may take M to be the Riemann sphere.) The equations of motion for the theory are

$$F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + [a_\mu, a_\nu] = 0. \tag{48}$$

The theory is best analyzed, for our purposes, in the gauge where a_0 is set to zero. In this gauge, the equations of motion (48) tell us that a_z, $a_{\bar{z}}$ are independent of time, but must satisfy the constraint

$$F_{\bar{z}z} \equiv \partial_{\bar{z}} a_z - \partial_z a_{\bar{z}} + [a_{\bar{z}}, a_z] = 0. \tag{49}$$

This constraint is just the Gauss law of the CS gauge theory. It can be solved for $a_{\bar{z}}$ as a function of a_z, at least as a power series in a_z. The result is

$$a_{\bar{z}} = \sum (-1)^{n-1} \int \frac{d^2 z_1}{\pi} \cdots \frac{d^2 z_n}{\pi} \, \frac{a_z(z_1, \bar{z}_1) a_z(z_2, \bar{z}_2) \ldots a_z(z_n, \bar{z}_n)}{(\bar{z} - \bar{z}_1)(\bar{z}_1 - \bar{z}_2) \ldots (\bar{z}_n - \bar{z})}. \tag{50}$$

This can be easily checked using $\partial_z(\frac{1}{\bar{z}-\bar{z}'}) = \pi \delta^{(2)}(z - z')$.

In the $a_0 = 0$ gauge, the action becomes

$$S = \frac{i\kappa}{\pi} \int dt d^2 x \; \mathrm{tr}(a_{\bar{z}} \partial_0 a_z). \tag{51}$$

This shows that $a_{\bar{z}}$ is essentially canonically conjugate to a_z. In fact in carrying out a variation of S, we find the surface term $\theta(t_f) - \theta(t_i)$, where

$$\theta = \frac{i\kappa}{\pi} \int_M d^2 x \; \mathrm{tr}(a_{\bar{z}} \delta a_z). \tag{52}$$

(We assume $a_{\bar{z}} \delta a_z$ to vanish at spatial infinity.) θ is the canonical one-form of the CS theory. (This is so by definition; the canonical one-form in any theory can be defined by θ, where the surface term in the variation of the action is $\theta_f - \theta_i$, the subscripts referring to the final and initial data surfaces.) θ is the analogue of $p_i dx^i$ of point-particle mechanics; for an action $S = \int dt dx \; [\frac{m\dot{x}^2}{2} - V(x)]$, we would find $\theta = m\dot{x}_i \delta x^i = p_i \delta x^i$. We can make another variation of θ, antisymmetrized with respect to the variation δa_z, denoted by the wedge product sign, and write

$$\begin{aligned} \omega \equiv \delta\theta &= \frac{i\kappa}{\pi} \int_M d^2 x \; \mathrm{tr}(\delta a_{\bar{z}} \wedge \delta a_z) \\ &= \tfrac{1}{2} \int d^2 x d^2 x' \; \omega_{AB}(x, x') \; \delta\xi^A(x) \wedge \delta\xi^B(x') \end{aligned} \tag{53}$$

where

$$\omega_{AB}(x, x') = -\frac{i\kappa}{2\pi} \begin{pmatrix} 0 & \delta(x - x')\delta^{ab} \\ -\delta(x - x')\delta^{ab} & 0 \end{pmatrix} \tag{54}$$

and $\delta\xi^A = (\delta a^a_{\bar{z}}, \delta a^a_z)$. ω defined by Eqs.(53,54) is called the symplectic structure and is of course the analogue of $dp_i dx^i$ of particle mechanics. The inverse of ω_{AB} gives the Poisson brackets, the commutators being i times the Poisson brackets. For our case we get

$$[\xi^A(x), \xi^B(x')] = i(\omega^{-1})^{AB}(x, x')$$

or

$$[a_{\bar{z}}^a(x), a_z^b(x)] = \frac{2\pi}{\kappa}\delta^{ab}\delta^{(2)}(x-x'). \tag{55}$$

One does not have to go through θ and ω to arrive at Eq.(55). One could simply use the fact that, from Eq.(51) the canonical momenta are $\pi = a_{\bar{z}}$, $\bar{\pi} = 0$. This is thus a constrained system in the Dirac sense and using the theory of constraints one can derive Eq.(55). The procedure of using $\omega_{AB}(x,x')$ is quicker.

In the expression (52) for θ, $a_{\bar{z}}$ is independent of a_z. We can however express $a_{\bar{z}}$ as a function of a_z via the constraint Eq.(49) or equivalently Eq.(50) and functionally integrate θ. In other words, we define $I(a_z)$ by

$$\delta I = \frac{i\kappa}{\pi}\int d^2x \ \mathrm{tr}[a_{\bar{z}}(a_z)\delta a_z]. \tag{56}$$

The solution for I is given by

$$I = i\kappa \sum \frac{(-1)^n}{n}\int \frac{d^2z_1}{\pi}\cdots\frac{d^2z_n}{\pi} \ \frac{\mathrm{tr}(a_z(z_1,\bar{z}_1)\ldots a_z(z_n,\bar{z}_n))}{\bar{z}_{12}\bar{z}_{23}\ldots\bar{z}_{n-1n}\bar{z}_{n1}}. \tag{57}$$

The quantity I has a rather simple interpretation. For one-dimensional point-particle mechanics, θ, as we mentioned earlier, is given by pdx. p is independent of x to begin with, but we can express it as a function of x via a constraint such as of fixed energy, e.g., $\frac{p^2}{2m} + V(x) = E$. Integral of $\theta = pdx$ then gives Hamilton's principal function or the eikonal, familiar as the exponent for the WKB wave functions of one-dimensional quantum mechanics. We have an analogous situation with Eq.(49) expressing $a_{\bar{z}}$ as a function of a_z. I is thus an eikonal of the CS theory [14,15].

I is in fact the Wess-Zumino-Novikov-Witten (WZNW) action [16]. We can write the gauge potential a_z as $a_z = -\partial_z U U^{-1}$ where U is in general not unitary; it is an $SL(N,\mathbf{C})$ matrix for gauge group $SU(N)$. Notice that since ∂_z has an inverse by virtue of $\partial_z \frac{1}{(\bar{z}-\bar{z}')} = \pi\delta^{(2)}(z-z')$, such a U can be constructed for any a_z, at least as a power series in a_z. I can then be written as $I = -i\kappa S_{WZNW}(U)$ where

$$S_{WZNW}(U) = \frac{1}{2\pi}\int_M d^2x \ \mathrm{tr}(\partial_z U\partial_{\bar{z}}U^{-1}) - \frac{i}{12\pi}\int_{M^3} d^3x \ \epsilon^{\mu\nu\alpha}\mathrm{tr}(U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\alpha U). \tag{58}$$

The second term, the so-called Wess-Zumino (WZ) term, involves an extension of U into a three-dimensional space. We take $M^3 = M \times [0,1]$ with $U(z,\bar{z},0) = 1$, $U(z,\bar{z},1) = U(z,\bar{z})$. The integrand of the WZ term becomes a total derivative once a (local) parametrization is chosen for U. This ensures that the physics is independent of small changes in how one makes the extension of U into the extra dimension. If we have two globally inequivalent extensions, say U_1 and U_2, the difference $S_{WZNW}(U_1) - S_{WZNW}(U_2) = 2\pi Q[g]$ where $Q[g]$ is the winding number of the map $g: S^3 \to G$; G is the space in which the matrices U take values, $SL(N,\mathbf{C})$ in the present case and $g = U_1$ on the upper hemisphere of S^3, which can be taken as one copy of M^3 and $g = U_2$ on the lower hemisphere, taken as another copy of M^3. Since Q is an integer, $e^{i\kappa S_{WZNW}}$ is independent of globally different extensions if κ is an integer [16].

The WZNW-action also obeys the following property, sometimes called the Polyakov-Wiegmann formula [17].

$$S(hU) \;=\; S(h) \;+\; S(U) \;-\; \frac{1}{\pi}\int_{M^2} \mathrm{Tr}(h^{-1}\partial_{\bar z}h\; \partial_z U\; U^{-1}) \tag{59}$$

The crucial point here is that in the cross-term h has only $\bar z$-derivative and U has only z-derivative. (They become ∂_-, ∂_+ in Minkowski space.)

The relationship of S_{WZNW} to I is easily seen by considering variations. Under the variation $U \to e^{\varphi}U \simeq (1+\varphi)U$, we find $\delta a_z = -D_z\varphi = -(\partial_z\varphi + [a_z,\varphi])$ and

$$\delta S_{WZNW} = \frac{1}{\pi}\int d^2x\; \mathrm{tr}(\partial_{\bar z}\varphi a_z). \tag{60}$$

Partially integrating and using $F_{\bar z z} = 0$ and $\delta a_z = -D_z\varphi$, we find that S_{WZNW} obeys Eq.(56) except for a factor $(-i\kappa)$, thus identifying $I = -i\kappa S_{WZNW}$.

Another quantity of interest is

$$K = -\frac{1}{\pi}\left[\,\kappa\int d^2x\; \mathrm{tr}(a_{\bar z}a_z) + i\pi I(a_z) + i\pi\tilde I(a_{\bar z})\right]. \tag{61}$$

It is easily checked that this is gauge-invariant. K has a nice interpretation as a Kähler potential, which I shall not go into here [1].

Finally notice that the eikonal I of Eq.(57) may be regarded as the expansion in powers of a_z of the logarithm of the functional determinant of $D_z = \partial_z + a_z$; i.e., $I = (-i\kappa)\log\det D_z = -i\kappa\;\mathrm{Tr}\log D_z$. The expression K of Eq.(61) is then given by $-\kappa\mathrm{Tr}\log(D_zD_{\bar z})$. The expansion of this expression in powers of the potential obviously gives I and $\tilde I$. The extra term $\int\frac{1}{\pi}\mathrm{tr}(a_{\bar z}a_z)$ is precisely the local counterterm needed to give a gauge invariantly regulated meaning to $\mathrm{Tr}\log(D_zD_{\bar z})$.

6. Solution for $\Gamma[A]$

We now return to Eqs.(45,46) for the quark-gluon plasma. We can rewrite Eq.(45) as

$$\delta W = 4\int d^4x\; \mathrm{tr}(a_{\bar z}\delta A_z) - \delta\int d^4x\; A^a_z A^a_z. \tag{62}$$

Comparing with Eq.(56), we see that, since $a_{\bar z}$, A_z obey the constraint Eq.(46), the solution is related to the eikonal I. The difference here is that A_z and $a_{\bar z}$ depend on all four coordinates x_μ, not just z, $\bar z$. However, there are no derivatives with respect to the transverse coordinates x^T in Eq.(46) and hence the solution for $a_{\bar z}$ in terms of a_z is the same as in Eq.(50), with a_z depending on x^T in addition to z, $\bar z$. The x^T-argument of all a_z factors is the same, i.e.,

$$a_{\bar z} = \sum(-1)^{n-1}\int\frac{d^2z_1}{\pi}\cdots\frac{d^2z_n}{\pi}\;\frac{A_z(z_1,\bar z_1,x^T)A_z(z_2,\bar z_2,x^T)\ldots A_z(z_n,\bar z_n,x^T)}{(\bar z-\bar z_1)\bar z_{12}\ldots\bar z_{n-1n}(\bar z_n-\bar z)}. \tag{63}$$

For the eikonal we get the same expression as Eq.(57) but with integration over the transverse coordinates, i.e.,

$$I = i\kappa\sum\frac{(-1)^n}{n}\int d^2x^T\;\frac{d^2z_1}{\pi}\cdots\frac{d^2z_n}{\pi}\;\frac{\mathrm{tr}(A_z(x_1)\ldots A_z(x_n))}{\bar z_{12}\bar z_{23}\ldots\bar z_{n1}}. \tag{64}$$

Since it is not relevant to the present discussion, we have, for the moment, set $\kappa = 1$. We then find the solution to Eq.(62) as

$$W = -4\pi i I(A_z) - \int d^4x \; A_z^a A_z^a. \tag{65}$$

(Strictly speaking $a_{\bar{z}}$ and A_z are not complex conjugates; the analogy holds better with a Chern-Simons theory of complex gauge group. However, we are only using the Chern-Simons analogy to obtain Eq.(65). The expression for $\Gamma[A]$ can also be directly checked to be a solution to Eqs.(45,46).)
From Eq.(39), Γ is given by

$$\Gamma = (N + \tfrac{1}{2}N_F)\frac{T^2}{12\pi}\left[\int d^4x \; 2\pi A_0^a A_0^a - \int d\Omega \{\int d^4x \quad A_z^a A_z^a + 4\pi i I(A_z)\}\right]. \tag{66}$$

We can now use the identity

$$\int d\Omega \; \mathrm{tr}(A_z A_{\bar{z}}) = -\tfrac{1}{2} \; \left[2\pi A_0^a A_0^a - \int d\Omega \; (A_z^a A_z^a)\right], \tag{67}$$

which follows by straightforward $d\Omega$-integration, to write

$$\Gamma = -(N + \tfrac{1}{2}N_F)\frac{T^2}{6\pi}\int d\Omega \left[\int d^4x \; \mathrm{tr}(A_z A_{\bar{z}}) + i\pi I(A_z) + i\pi \tilde{I}(A_{\bar{z}})\right] \tag{68a}$$

$$= (N + \tfrac{1}{2}N_F)\frac{T^2}{6}\int d\Omega \; K(A_z, A_{\bar{z}}) \tag{68b}$$

where K is given by Eq.(61) with $\kappa = 1$ and with the additional integration over coordinates $\vec{x}^T$ transverse to $\vec{Q}$. If we write $A_z = -\partial_z U U^{-1}$ and $A_{\bar{z}} = {U^{\dagger}}^{-1}\partial_{\bar{z}} U$, we can write Eq.(68) in terms of S_{WZNW}, using Eq.(59), as

$$\Gamma = -(N + \tfrac{1}{2}N_F)\frac{T^2}{6} S_{WZNW}(U^{\dagger}U). \tag{69}$$

(Again, a suitable additional integration over the transverse coordinates is understood.)

From Eqs.(41,42), it may seem that our solution is ambiguous up to the addition of a purely gauge invariant term. But for hard thermal loops, the additional structure that W depends only on A_z tells us that the gauge invariant piece must obey $D_z \frac{\delta W}{\delta A_z} = 0$. Since ∂_z is invertible, at least perturbatively, there is no nontrivial solution to this equation. Thus Eq.(66) or Eq.(68) is the unique solution and Γ so defined must indeed be the generator of hard thermal loops.

In the last section, we noted that $K(A_z, A_{\bar{z}})$ can be considered as $\mathrm{Trlog}(D_z D_{\bar{z}})$. Since D_z and $D_{\bar{z}}$ are the chiral Dirac operators in two dimensions, $\mathrm{Trlog}(D_z D_{\bar{z}})$ is clearly the photon mass term of the Schwinger model, for Abelian gauge fields. More generally, for non-Abelian fields as well, we can consider $\mathrm{Trlog}(D_z D_{\bar{z}})$ as a gauge-invariant mass term. It is perhaps fitting that the gauge-invariant Debye screening mass term in four-dimensional QCD is given by suitable integrations of

such a two-dimensional mass term (with, of course, the additional x^T-dependence). Equally appropriately, the Chern-Simons term, from which we derive $K(A_z, A_{\bar{z}})$, is also the mass term for gauge fields in three dimensions.

We close this section with two remarks on Γ. The CS theory of Eq.(47) violates parity. We see that this parity violation disappears, as indeed it should, by integration over the orientations of $\vec{Q}$, for the QCD case. Alternatively, expression (68a) is manifestly parity-symmetric with $A_z \leftrightarrow A_{\bar{z}}$ (or $A_+ \leftrightarrow A_-$ in Minkowski space), $Q \leftrightarrow Q'$ under parity. Secondly, instead of setting $\kappa = 1$ and then having a prefactor $(N + \frac{1}{2}N_F)\frac{T^2}{6}$ in Eq.(68), we could simply choose $\kappa = (N + \frac{1}{2}N_F)\frac{T^2}{6}$. For WZNW actions, if U has a non-Abelian unitary part, any action we use must be an integer times S_{WZNW}, as we have already seen. In our case, Γ involves $S_{WZNW}(U^\dagger U)$ or $S_{WZNW}(H)$ where H is hermitian. The third homotopy group is trivial for the space of hermitian matrices and so, as expected, there is no argument for quantization of the coefficient, which is $(N + \frac{1}{2}N_F)\frac{T^2}{6}$ for us.

7. Continuation to Minkowski space

We shall now consider how the above results can be continued back to Minkowski space [2]. Evidently, we expect $A_z \to A_+,\ A_{\bar{z}} \to A_-$ and $\partial_z \to \partial_+,\ \partial_{\bar{z}} \to \partial_-$. The inverses of $\partial_\pm$, however, require specification of boundary conditions or $i\epsilon$-prescriptions. This can be easily seen in momentum space where the inverses have denominators like $k\cdot Q$ which can vanish; one must give a prescription on how these singularities are to be handled for the k^0-integration. The appropriate $i\epsilon$-prescription depends on the physical context and the quantity being considered. Very often, one is interested in the evolution of fields in a plasma, which can be described by the operator field equations. For this case, the retarded Green's functions are the appropriate ones. Let me illustrate this with a simple example, viz., a plasma in electrodynamics. The field equations can be written as [18]

$$\partial^2 A^\mu \;=\; iS^{-1}\frac{\delta S}{\delta A^{in}_\mu} \equiv J^\mu \tag{70}$$

(We have chosen the Feynman gauge for simplicity.) S is the scattering operator considered as a function of the incoming field A^{in}_μ. We can expand the current as a function of the field A^{in}_μ, obtaining to linear order in the field,

$$J^\mu = j^\mu - i\int d^4y\ \Theta(x^0 - y^0)[j^\mu(x), j^\nu(y)]A^{in}_\nu(y) \tag{71}$$

where j^μ is the current in the interaction picture, say $\bar{q}_I\gamma^\mu q_I$ in terms of the fernion field q_I in the interaction picture. Notice that the we have the retarded commutator of the currents j^μ. Eq.(71) is an operator equation which is generally true. We can take any kind of expectation value of this equation; the thermal result which is the induced current in the plasma is obtained by taking a thermal average. (The result is the Kubo formula.) To the order we are interested in, the incoming and

interacting fields can be considered the same and Eq.(70) gives its evolution. The appropriate boundary condition for the induced current which governs the evolution of the fields, we see, is the retarded condition. Of course, the induced current can also be calculated in terms of the induced or effective action Γ as $-\frac{\delta\Gamma}{\delta A_\mu}$. The strategy for continuation to Minkowski space is thus to calculate the induced current in Euclidean space and then to use $i\epsilon$'s appropriately in denominators involving $k\cdot Q$'s so as to get the retarded condition. If J^μ is expanded to higher orders in the field, we encounter multiple retarded commutators. The $i\epsilon$'s must be inserted appropriately so that we get this retardation structure. With this understanding, we can write the equations for the evolution of fields of soft momenta in the quark-gluon plasma as

$$D_\nu F^{\nu\mu,a} = J^{\mu,a} \tag{72a}$$

$$J^{\mu,a} = \sum_1^\infty \int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4} e^{i(\sum k)\cdot x} J_n^{\mu,a}(k) \tag{72b}$$

$$\begin{aligned} J_n^{\mu,a}(k) = \frac{\kappa}{\pi}\int d\Omega\Bigg[&\mathrm{Tr}\left((\frac{-it^aQ^\mu}{2})A_-(k_1) + A_+(k_1)(\frac{-it^aQ'^\mu}{2})\right)\delta_{n,1} \\ &+\left\{-(2i)^{n-1}\mathrm{Tr}\left((\frac{-it^aQ^\mu}{2})A_+(k_1)\cdots A_+(k_n)\right)F(k_1,\cdots k_n) + (Q\leftrightarrow Q')\right\}\Bigg] \end{aligned} \tag{72c}$$

where

$$F(k_1,\cdots k_n) = \sum_{i=0}^{n}\frac{q_i}{(\bar q_i-\bar q_0)(\bar q_i-\bar q_1)\cdots(\bar q_i-\bar q_{i-1})(\bar q_i-\bar q_{i+1})\cdots(\bar q_i-\bar q_n)} \tag{73a}$$

$$\bar q_i = \sum_{j=1}^{i}(k_j\cdot Q - i\epsilon_j), \qquad q_i = \sum_{j=1}^{i} k_j\cdot Q' \tag{73b}$$

We may reexpress the current as

$$J^{\nu a} = -\frac{\kappa}{2\pi}\int d\Omega\ \mathrm{Tr}\left[(-it^a)\left\{(a_+ - A_+)Q'^\nu + (Q'\leftrightarrow Q)\right\}\right] \tag{74}$$

where a_+ is defined by

$$\partial_- a_+ - \partial_+ A_- + [A_-, a_+] = 0 \tag{75}$$

The retarded condition is to be used in solving Eq.(75).

It is easy to check that the imaginary part of the current given by these equations gives the Landau damping effects. To linear order in the field A^a_μ, we have $J^a_\mu = \int d^4y\ \Pi^{ab}_{\mu\nu}(x,y)A^{\nu a}(y)$, with

$$\begin{aligned} \Pi^{ab}_{\mu\nu}(x,y) &= \delta^{ab}\int\frac{d^4k}{(2\pi)^4}\ e^{ik(x-y)}\ \Pi_{\mu\nu}(k) \\ \Pi_{\mu\nu}(k) &= -\frac{\kappa}{2\pi}\left[4\pi g_{\mu 0}g_{\nu 0} - k_0\int d\Omega\ \frac{Q_\mu Q_\nu}{k\cdot Q - i\epsilon}\right] \end{aligned} \tag{76}$$

The imaginary part exists only for spacelike momenta and is evidently given by

$$
\begin{aligned}
\text{Im } \Pi_{\mu\nu}(k) \; &= k^0 \; \frac{\kappa}{2\pi} P_{\mu\nu} \\
P_{\mu\nu} &\equiv \int d\Omega \; Q_\mu Q_\nu \delta(k \cdot Q) \\
&= -k^2 \Theta(-k^2) \frac{6\pi}{|\vec{k}|^2} \left[\frac{1}{3} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \tfrac{1}{2} \tilde{P}_{\mu\nu} \right] \\
\tilde{P}_{0\nu} &= \tilde{P}_{\mu 0} = 0 \\
\tilde{P}_{ij} &= \delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}
\end{aligned}
\tag{77}
$$

As I said before, to the accuracy we are interested in here, the distinction between the interacting and incoming field is irrelevant. One can, of course, go beyond this and include higher order corrections, using the time-contour approach due to Schwinger and Bakshi and Mahanthappa, which was rediscovered by Keldysh [19]. The operator approach is conceptually clearer in indicating why the retarded condition is appropriate, but a functional rewriting is useful in going to higher orders. We define

$$
Z[\eta] = \frac{Tr\rho \; \mathcal{T}_C \exp(iS_{int} + iA \cdot \eta)}{Tr\rho} \tag{78}
$$

where the time-integral goes from $-\infty$ to ∞, folds back and goes from ∞ to $-\infty$; $\mathcal{T}_C$ denotes ordering along this time-contour. ρ is the thermal density matrix and η^μ is a source function. One can represent $Z[\eta]$ as a functional integral

$$
Z[\eta] = \int d\mu(A, c, \bar{c}) \exp(iS_C(A, c, \bar{c}) + iA_\mu \eta^\mu) \tag{79}
$$

The action is again defined on the time-contour. The Green's functions which arise in perturbatively integrating out the fields are to be taken as the real-time thermal Green's functions, as in Eqs.(4,9). In this time-contour version, the field is given by is

$$
< A_\mu > = \frac{-i}{Z} \frac{\delta Z}{\delta \eta^\mu(x)} \tag{80}
$$

We differentiate with respect to η on the first branch for the time-contour. It is evident that we can now define a generator for the one-particle-irreducible graphs $\Gamma_C[A]$, such that $\frac{\delta \Gamma_C}{\delta A} = -\eta$. The evolution of a field configuration in the medium is given by $\frac{\delta \Gamma_C}{\delta A} = 0$; we can now set the source to zero. Separating out the term $\int (\partial_\nu A_\mu)^2$, which gives $\partial^2 A_\mu$, we get the current as

$$
< J^a_\mu > = \frac{\delta \Gamma^*_C}{\delta A^a_\mu} \tag{81}
$$

$\Gamma_C = \int_C \frac{1}{2} (\partial_\nu A_\mu)^2 + \Gamma^*_C$. Eq.(81) is the same as the expectation value of our operator definition, with one-particle reducible terms summed up and expressed in terms of $< A^a_\mu >$ rather than A^{ain}_μ. In writing out Eq.(81), we encounter A's on the second branch of time. From the operator definition, we see that fields on the second branch

are the conjugates of fields on the first branch. (In the usual time-contour approach, one considers equations for the two-point functions, here we consider equations for the fields; this is the only difference.)

To summarize, the functional transcription of the operator analysis gives the following. The evolution of fields in the medium is given by $\frac{\delta\Gamma_C^*}{\delta A}=0$, with fields on the second branch being conjugates of fields on the first branch. One can check that this gives the retarded prescription we use. For example, upto linear order in the field A_μ, we get,

$$\begin{aligned}\partial^2 A_\mu &= -i\int_C d^4y \;\; < T_C j_\mu(x) j_\nu(y) > A^\nu(y)\\ &= -i\left(\int_{-\infty}^{\infty} < T(j_\mu(x)j_\nu(y)) > + \int_{\infty}^{-\infty} < j_\nu(y) j_\mu(x) >\right) A^\nu(y)\,,\\ &= -i\int \theta(x^0-y^0) < [j_\mu(x), j_\nu(y)] > A^\nu(y)\end{aligned} \tag{82}$$

which agrees with Eq.(71).

8. Derivation of $\Gamma[A]$ from the Boltzmann equation

The equations of motion we obtained and the expression for the induced current have also been derived using suitable truncations of the Schwinger-Dyson equations [20], from effective actions for composite operators [21] and from kinetic theory [22]. These are all related approaches; here I shall briefly indicate the kinetic theory approach. The classical equations of motion for non-Abelian particles, the so-called Wong equations [23], are

$$m\frac{dx^\mu}{d\tau} = p^\mu \tag{83a}$$

$$m\frac{dp^\mu}{d\tau} = gq^a F_a^{\mu\nu} p_\nu \tag{83b}$$

$$m\frac{dq^a}{d\tau} = -gf^{abc}(p^\mu A_\mu^b)q^c \tag{83c}$$

Here q^a represents the classical color charge of the particle. (The color degrees of freedom can also be described in a phase space way; the appropriate space is the Lie group modulo the maximal torus [24].)

For particles obeying the Wong equations, the collisionless Boltzmann equation for the distribution function $f(x,p,q)$ is given by [25]

$$p^\mu\left[\frac{\partial}{\partial x^\mu} - gq_a F^a_{\mu\nu}\frac{\partial}{\partial p_\nu} - gf_{abc}A^b_\mu q^c \frac{\partial}{\partial q_a}\right] f(x,p,q) = 0 \tag{84}$$

The Boltzmann equation is invariant under gauge transformations $gA_\mu \to UgA_\mu U^{-1} - U\partial_\mu U^{-1}$, $f(x,p,q)\to f(x,p,UqU^{-1})$. We then seek a perturbative solution of the form $f = f^{(0)} + gf^{(1)} + ...$, where $f^{(0)} = n_p$ is the equilibrium distribution, appropriately chosen for bosons and fermions. The Boltzmann equation, to the first order, gives

$$p^\mu\left[\frac{\partial}{\partial x^\mu} - gf^{abc}A^b_\mu q_c\frac{\partial}{\partial q^a}\right] f^{(1)} = p^\mu q_a F^a_{\mu\nu}\frac{\partial}{\partial p_\nu} f^{(0)} \tag{85}$$

The color current, to leading order, is

$$J^a_\mu = g^2 \int [dq]\ p_\mu q^a f^{(1)} \tag{86}$$

$[dq]$ is a measure for integration over the color charges which can be constructed in terms of the group coordinates modulo the maximal torus. By virtue of Eq.(85), J^a_μ satisfies the equation

$$(p \cdot DJ^\mu)^a = g^2 p^\mu p^\nu F^b_{\nu\alpha} \frac{\partial}{\partial p_\alpha} \left(\int dq\ q^a q_b f^{(0)} \right) \tag{87}$$

We now integrate this equation over the magnitude of $\vec{p}$. Defining

$$\begin{aligned} J^\mu &= \int d\Omega\ \mathcal{J}^\mu \\ \mathcal{J}^\mu(x,Q) &= \int \frac{d|\vec{p}|dp_0}{(2\pi)^3}\ 2\Theta(p_0)\delta(p^2)|\vec{p}|^2\ J^\mu \\ &\equiv \frac{\delta W}{\delta A_\mu} - \frac{\kappa}{2\pi} Q^\mu A_0, \end{aligned} \tag{88}$$

we see that Eq.(87) for the current becomes the zero-curvature condition Eq.(75); the rest of the analysis is as before, leading to Eqs.(72,73). This derivation from kinetic theory shows that the hard thermal loops are classical; they include thermal fluctuations, not quantum fluctuations. Of course, we have already seen this from another point of view, viz., the thermal part of the propagator is on-shell and so, the T-dependent part of a one-loop diagram describes the tree-level absorption and emission of particles from the heat bath.

9. An auxiliary field and Hamiltonian analysis

We now turn to the question of how we can use the effective action $\Gamma[A]$. There are two related but different contexts in which we need the expression for $\Gamma[A]$. The first is in setting up thermal perturbation theory. The version of $\Gamma[A]$ as given in Eqs.(64,68) is probably best suited for this purpose. The resummed perturbation theory can be set up as follows. We introduce a splitting of the Yang-Mills action as

$$\begin{aligned} S &= S_0 - c\Gamma \\ S_0 &= \int d^4x\ \left(-\tfrac{1}{4}F^2 \right)\ + \Gamma \end{aligned} \tag{89}$$

We define propagators and vertices and start off the perturbative expansion using S_0. $c\Gamma$ will be treated as a 'counterterm', nominally one order higher in the thermal loops than S_0. Eventually of course c is taken to be 1, so that we are only achieving a rearrangement of terms in the perturbative expansion. (As usual we must have gauge fixing and ghost terms.) Using this procedure one can calculate quantities

which require resummations such as the gluon decay rate in the plasma [9]. In Eq.(89) we have not displayed the quark terms. In addition to the quark kinetic energy terms, there are hard thermal loops which give a T-dependent mass to the quarks [12,26]. We have not discussed this so far, since Chern-Simons theory does not give any new insights on this question. This mass term is actually given by

$$
\begin{aligned}
\Gamma^{(q)}[q, A] &= \int d^4x d^4y \ \bar{q}(x)\Sigma(x,y)q(y) \\
\Sigma(x,y) &= \frac{g^2T^2C_F}{64\pi}\int d\Omega \ \gamma\cdot Q \ F(x,y) \\
(iQ\cdot D)F(x,y) &= \delta^{(4)}(x-y)
\end{aligned}
\tag{90}
$$

where $C_F = t^a t^a$ is the value of the quadratic Casimir for the quark representation. This action must also be added to and subtracted from the QCD action and treated in a manner analogous to $\Gamma[A]$ to set up the perturbation theory properly.

In addition to its role in rearrangement of perturbation theory, we can also use $\Gamma[A]$ added to the usual Yang-Mills action, viz.,

$$
S_{eff} = \int d^4x \ \left(-\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a\right) \ + \Gamma[A], \tag{91}
$$

as an effective action for the soft modes. (This is the spirit of Eqs.(72,73).) The nonlocality of $\Gamma[A]$ makes it somewhat difficult to handle in this context. It is useful to rewrite $\Gamma[A]$ using an auxiliary field which makes the equations of motion local [3]. The auxiliary field is also useful in setting up a Hamitonian analysis of the dynamics of the soft modes. The auxiliary field we use will be an $SU(N)$-matrix field $G(x,\vec{Q})$ which is a function of x and $\vec{Q}$, i.e., defined on $\mathcal{M}^4\times S^2$, $\mathcal{M}^4$ being Minkowski space. Further $G(x,\vec{Q})$ must satisfy the condition $G^\dagger(x,\vec{Q}) = G(x,-\vec{Q})$. The action is given by

$$
\begin{aligned}
\mathcal{S} = \int -\tfrac{1}{4}F^2 \ + \kappa\int d\Omega \ \Bigg[d^2x^T \ \mathcal{S}_{WZNW}(G) \ + \ \frac{1}{\pi}\int d^4x \ \mathrm{Tr}(G^{-1}\partial_- G \ A_+ \\
- \ A_-\partial_+ G \ G^{-1} + A_+G^{-1}A_-G - A_+A_-) \Bigg]
\end{aligned}
\tag{92}
$$

where $\mathcal{S}_{WZNW}(G)$ is the WZNW action of Eq.(58) with U replaced by G. The quantity in the square brackets in Eq.(92) is the gauged WZNW action [27]. It is invariant under gauge transformations with G transforming as $G\to G' = h(x)G \ h^{-1}(x), \ h(x)\in SU(N)$. The equations of motion for the action (92) are

$$
\partial_+A_- - \partial_- a_+ \ + \ [a_+, A_-] = 0 \tag{93a}
$$

$$
a_+ \equiv GA_+G^{-1} - \partial_+G \ G^{-1} \tag{93b}
$$

$$
(D_\mu F^{\mu\nu})^a - J^{\nu a} = 0 \tag{94a}
$$

$$J^{\nu a} = -\frac{\kappa}{2\pi}\int d\Omega\ \mathrm{Tr}[\{(-it^a)(a_+ - A_+)Q'^{\nu}\} + (Q' \leftrightarrow Q)] \tag{94b}$$

$$= -\frac{\kappa}{2\pi}\int d\Omega\ \mathrm{Tr}[(-it^a)\{G^{-1}D_-G\ Q^{\nu}\ -\ D_+G\ G^{-1}Q'^{\nu}\}] \tag{94c}$$

Clearly these equations are equivalent to Eqs.(72,74,75); the only difference is that the equation defining a_+ in Eq.(74), viz. Eq.(75), is now obtained as the equation of motion (93a) for G. Notice that the current in Eq.(94c) looks like the current of a matter field; thus except for the fact that G depends on $\vec{Q}$, QCD, with hard thermal loops added, is no stranger than Yang-Mills theory coupled to a matter field. The equations of motion for G has no independent solutions, so that we are not changing the physical degrees of freedom by introducing the auxiliary field. This can be seen as follows. We can parametrize the potentials $A_\pm$ in terms an $SU(N)$-matrix $V(x,\vec{Q})$ as

$$A_+ =\ -\partial_+ V\ V^{-1}, \qquad\qquad A_- =\ -\partial_- V'\ V'^{-1} \tag{95}$$

where $V'(x,\vec{Q}) = V(x,-\vec{Q})$. The general solution to Eq.(93) is then given by

$$G(x,\vec{Q})\ =\ V(x,-\vec{Q})B(x^+,x^T,\vec{Q})C(x^-,x^T,\vec{Q})V^{-1}(x,\vec{Q}) \tag{96}$$

where C is an arbitrary $SU(N)$-matrix depending on the variables indicated and B is given by C with $\vec{Q} \to -\vec{Q}$. The matrices B, C represent the new or independent degrees of freedom for the field G. Notice, however, that the parametrization (95) of the potentials has redundant variables; a transformation $V \to VU(x^-,x^T,\vec{Q})$, with a corresponding change in V', leaves the potentials invariant. Further, ordinary gauge transformations act on the matrix V as $V \to h(x)V$. Thus the physical subspace for the components $A_\pm$ is given in terms of the matrices V with the identifications

$$V(x,\vec{Q}) \sim h(x)V(x,\vec{Q})U(x^-,x^T,\vec{Q}) \tag{97}$$

The gauge freedom of multiplying V's by matrices which do not depend on x^+, viz., U's in Eq.(97), shows that the we can reduce G to just $V(x,\vec{Q})V^{-1}(x,\vec{Q})$. There are thus no new real dynamical degrees of freedom in G. The action can be simplified for this G to

$$\Gamma[A,G] =\ -\kappa\int d\Omega\ d^2x^T\ S_{WZNW}\big(V^{-1}(x,-\vec{Q})V(x,\vec{Q})\big) \tag{98}$$

This agrees with the expression obtained by Feynman graph evaluation of the hard thermal loops.

The Hamiltonian for the effective action (92) can be obtained by following the usual analysis for the WZNW-action; it is given by [3]

$$\mathcal{H} = \int d^3x\ \left\{\frac{E^2+B^2}{2}\ +\ \frac{\kappa}{8\pi}\int d\Omega\ \mathrm{Tr}\left[(D_0G\ D_0G^{-1}) + (\vec{Q}\cdot\vec{D}G\ \vec{Q}\cdot\vec{D}G^{-1}\right] - A_0^a\mathcal{G}^a\right\} \tag{99}$$

where $F^a_{0i} = E^a_i,\ F^a_{ij} = \epsilon_{ijk}B^a_k$ and

$$\mathcal{G}^a =\ (\vec{D}\cdot\vec{E})^a\ +\ \frac{\kappa}{2\pi}\int d\Omega\ \mathrm{Tr}\big[(-it^a)(G^{-1}D_-G\ -\ D_+G\ G^{-1})\big] \tag{100}$$

$\mathcal{G}^a = 0$ is the Gauss law of the theory; it is also the time component of the equation of motion for the gauge field. Expression (99) makes it clear that the Hamiltonian is positive for all configurations which are physical, i.e., obey the Gauss law. (It may be worth recalling that, even for the Maxwell theory, the canonical Hamiltonian is positive only for physical configurations.) We find, comfortingly, that the effective theory for the soft modes has positive energy.

The Hamiltonian analysis is most easily carried out, as in the usual cases, in the gauge $A_0^a = 0$. We start by defining the currents

$$\begin{aligned} J_+ \;&=\; \frac{\kappa}{4\pi} D_+ G \; G^{-1} \;=\; (-it^a) J_+^a \\ J_- \;&=\; -\frac{\kappa}{4\pi} G^{-1} D_- G \;=\; (-it^a) J_-^a \end{aligned} \tag{101}$$

By virtue of the property $G^{-1}(x, \vec{Q}) = G(x, -\vec{Q})$, these are related by $J_+(x, -\vec{Q}) = J_-(x, \vec{Q})$. The Hamiltonian can be written in terms of these currents as

$$\mathcal{H} = \int d^3x \; \left\{ \frac{E^2 + B^2}{2} \; + \frac{2\pi}{\kappa} \int d\Omega \; (J_+^a J_+^a \; + \; J_-^a J_-^a) \right\} \tag{102}$$

We have chosen the $A_0^a = 0$ gauge; Gauss law must henceforth be imposed as a constraint. (For fixed $\vec{Q}$, the integrand of the second term involving the square of J_+ and the square of J_- is the Sugawara form of the Hamiltonian, well-known in the context of two-dimensional current algebras.) The equations of motion in the $A_0^a = 0$ gauge are

$$E_i^a \; = \partial_0 A_i^a \tag{103a}$$

$$\partial_0 E_i^a \; + \; \epsilon_{ijk} (D_j B_k)^a \;=\; \int d\Omega \; Q_i (J_+^a \; - \; J_-^a) \tag{103b}$$

$$(D_- J_+)^a \;=\; -\frac{\kappa}{8\pi} E_i^a Q_i \tag{103c}$$

Equation (103a) is just the definition of the electric field; however, in a Hamiltonian approach, it is an equation of motion and we have displayed it as such.

The commutation rules must be such that Eqs.(103) follow as the Heisenberg equations of motion for the Hamiltonian (102). Knowing the current algebra of the WZNW-theory, we can make a guess as to what the appropriate commutation rules are for our problem and verify them by checking that they lead to Eqs.(103) starting from the Hamiltonian (102). The commutation rules are then seen to be

$$[\; E_i^a(\vec{x}), A_j^b(\vec{x}') \;] \; = -i\delta^{ab}\delta_{ij}\delta(\vec{x} - \vec{x}') \tag{104a}$$

$$[\; E_i^a(\vec{x}), J_\pm^b(\vec{x}') \;] \; = \pm i \frac{\kappa}{4\pi} Q_i \delta^{ab} \delta(\vec{x} - \vec{x}') \tag{104b}$$

$$\begin{aligned} [\; J_\pm^a(\vec{x}, \vec{Q}), J_\pm^b(\vec{x}', \vec{Q}') \;] \; &= i f^{abc} J_\pm^c \delta(\vec{x} - \vec{x}') \delta(\vec{Q}, \vec{Q}') \\ &\mp \frac{\kappa}{4\pi} Q_i (D_x)_i^{ab} \delta(\vec{x} - \vec{x}') \delta(\vec{Q}, \vec{Q}') \end{aligned} \tag{104c}$$

$$[\; J_+^a(\vec{x}, \vec{Q}), J_-^b(\vec{x}', \vec{Q}') \;] \; = 0 \tag{104d}$$

All other commutators vanish. $\delta(\vec{Q},\vec{Q}')$ stands for the δ-function on the sphere corresponding to the unit vector $\vec{Q}$, i.e., $\int d\Omega_{Q'}\ \delta(\vec{Q},\vec{Q}')f(\vec{Q}') = f(\vec{Q})$.

The algebra (104) is an interesting extension of the usual WZNW-current algebra; we are in four dimensions and have a gauge field as well. We can check that the commutation rules (104) obey the Jacobi identity. Since we are postulating the algebra, this is a necessary check. Of course, commutation rules can also be obtained from the action by standard quantization procedures [3]. The condition $J_+^a(x-\vec{Q})\ -\ J_-^a(x,\vec{Q}) = 0$ has to be imposed as a constraint, just like the Gauss law.

10. Plasma waves

Long wavelength and low frequency plasma waves are the classical solutions of the effective theory Eq.(91). It also includes effects such as the screening of Coulomb fields. These features can be seen by examining the Abelian case or electrodynamics. In this case, the terms in Γ which are cubic or higher order in A_μ are zero and in terms of the Fourier components of A_μ, we can write

$$S_{eff} = \tfrac{1}{2}\int \frac{d^4k}{(2\pi)^4}\ A_\mu(-k)M^{\mu\nu}(k)A_\nu(k) \tag{105a}$$

where

$$M^{\mu\nu} = (-k^2g^{\mu\nu}+k^\mu k^\nu) + \frac{\kappa}{2\pi}\left[4\pi g^{\mu0}g^{\nu0} - k^0\int d\Omega\ \frac{Q^\mu Q^\nu}{k\cdot Q}\right]. \tag{105b}$$

$k^\mu M_{\mu\nu} = 0$ in accordance with the requirement of gauge invariance. $\kappa = N_F e^2T^2/6$ for electrodynamics; we have restored the coupling constant e in this expression. For the plasma waves, we will see that the wavevector is timelike; hence the imaginary part of $M_{\mu\nu}$ is irrelevant.

We can split A_μ into a gauge dependent part and gauge invariant components as

$$A_\mu = k_\mu\Lambda(k) + \alpha_\mu + \beta_\mu \tag{106}$$

where Λ shifts under gauge transformations and α_μ, β_μ are gauge invariant. We take

$$\alpha_0 = (\frac{\vec{k}^2}{\vec{k}^2-k_0^2})\phi \qquad \alpha_i = \frac{k_0}{\sqrt{\vec{k}^2}}e_i^{(3)}\ (\frac{\vec{k}^2}{\vec{k}^2-k_0^2})\phi$$

$$\beta_0 = 0 \qquad \beta_i = e_i^{(\lambda)}a_\lambda, \qquad \lambda = 1,2. \tag{107}$$

(Here we are considering fields off-shell and so $k^0 \neq |\vec{k}|$.) The e_i's form a triad of spatial unit vectors which may be taken as

$$e_i^{(3)} = \frac{k_i}{\sqrt{\vec{k}^2}}, \qquad i = 1,2,3,$$

$$e^{(1)} = (\epsilon_{ij}\frac{k_j}{\sqrt{k_T^2}},0), \qquad e^{(2)} = (\frac{k_3k_i}{\sqrt{k_T^2\vec{k}^2}},-\sqrt{\frac{k_T^2}{\vec{k}^2}}), \qquad i=1,2, \tag{108}$$

where $k_T^2 = k_1^2 + k_2^2$. Notice that $k_i e_i^{(\lambda)} = 0,\ \lambda = 1,2$. ϕ and a_λ are the gauge invariant degrees of freedom in Eq.(106). When the mode decomposition (106) is used in Eq.(105) we get

$$S_{eff} = \tfrac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[\ \beta_i(-k)(\frac{\vec{k}^2\delta_{ij} - k_i k_j}{\vec{k}^2})M^T(k)\beta_j(k) + \phi(-k)M^L(k)\phi(k)\right] + \int \phi J^0 + \beta_i J^i \quad (109)$$

where

$$\begin{aligned} M^T(k) &= k_0^2 - \vec{k}^2 - \Omega_T^2(k_0,\vec{k}) \\ \Omega_T^2 &= \kappa\left[\ \frac{k_0^2}{\vec{k}^2} + (1-\frac{k_0^2}{\vec{k}^2})\frac{k_0}{2|\vec{k}|}L\ \right] \\ M^L(k) &= \frac{\vec{k}^2}{(k_0^2-\vec{k}^2)}\left(k_0^2 - \vec{k}^2 - \Omega_L^2\right) \\ \Omega_L^2 &= 2\kappa\ \left(\frac{k_0^2-\vec{k}^2}{\vec{k}^2}\right)(\frac{k_0}{2|\vec{k}|}L\ \ -1) \end{aligned} \quad (110)$$

$$L = \log(\frac{k_0 + |\vec{k}|}{k_0 - |\vec{k}|}). \quad (111)$$

We have also included an interaction term with a conserved source J_μ in Eq.(109); i.e., we include $\int A_\mu J^\mu$ and simplify it using Eq.(106). From Eq.(108) we see that the interaction between charges in the plasma is governed by $(M^L)^{-1}$ which shows the Debye screening with a Debye mass $m_D = \sqrt{2\kappa}$. The action (109) can also give free wavelike solutions. The dispersion rules for these plasma waves would be $M^T = 0$ for the transverse waves and $(\frac{\vec{k}^2-k_0^2}{\vec{k}^2})M^L = 0$ for the longitudinal waves [28]. (The extra factor multiplying M_L is from rewriting $\phi(k)$ in terms of the potential.)

Non-Abelian plasma waves can be defined in a similar way as propagating solutions to the equations of motion given by Eq.(91). For plasma waves in Abelian subgroups, only terms in Γ upto the quadratic order in A_μ are important; the analysis is the same as for the Abelian plasma waves with $m_D^2 = 2\kappa = (N + \frac{1}{2}N_F)\frac{g^2T^2}{3}$ and $\kappa = (N+\frac{1}{2}N_F)\frac{g^2T^2}{6}$ in $M^T(k)$.

It is interesting to consider these waves also from the Hamiltonian point of view. We can choose $A_0 = 0$ gauge and introduce the parametrization

$$A_i = e_i^{(3)}a + e_i^{(\lambda)}a_\lambda \quad (112a)$$

$$E_i = e_i^{(3)}\Pi + e_i^{(\lambda)}\Pi_\lambda \quad (112b)$$

We have dropped the color indices, since we are in an Abelian subgroup. The Hamiltonian becomes

$$\mathcal{H} = \int d^3x\ \tfrac{1}{2}\left[\Pi^2 + \Pi_\lambda^2 + \partial_j a_\lambda \partial_j a_\lambda + \frac{4\pi}{\kappa}\int d\Omega(J_+^2 + J_-^2)\right] \quad (113)$$

The commutation rules are

$$[\Pi(\vec{x}), a(\vec{y})] = -i\delta(\vec{x}-\vec{y}), \qquad [\Pi_\lambda(\vec{x}), a'_\lambda(\vec{y})] = -i\delta_{\lambda\lambda'}\delta(\vec{x}-\vec{y}) \quad (114a)$$

$$[\Pi(\vec{x}), J_\pm(\vec{y})] = \pm i\frac{\kappa}{4\pi}\vec{Q}\cdot e^{(3)}\delta(\vec{x}-\vec{y}), \qquad [\Pi_\lambda(\vec{x}), J_\pm(\vec{y})] = \pm i\frac{\kappa}{4\pi}\vec{Q}\cdot e^{(\lambda)}\delta(\vec{x}-\vec{y}) \quad (114b)$$

The creation operators for the eigenstates obey the eigenvalue equation

$$[\mathcal{H}, \alpha] = \omega \alpha \tag{115}$$

For the longitudinal modes, we can take

$$\alpha = \int d^3x \ e^{-i\vec{k}\cdot\vec{x}} \left[c_1 \Pi + c_2 a + \int d\Omega (c_3 J_+ + c_4 J_-) \right] \tag{116}$$

Using this in the eigenvalue equation, we find

$$\omega \ c_1 = i \frac{\kappa}{4\pi} \int d\Omega \ \vec{Q} \cdot e^{(3)} (c_3 - c_4) \tag{117a}$$

$$c_3 = -\frac{i}{|\vec{k}|} \frac{\vec{k}\cdot\vec{Q}}{(\omega + \vec{k}\cdot\vec{Q})} \ c_1 \tag{117b}$$

and $c_2 = 0, \ c_4 = c_3(-\vec{Q})$. Using Eq.(117b) in Eq.(117a), we find that a nonzero solution requires

$$\omega - \frac{\kappa}{2\pi} \int d\Omega \ \frac{(\vec{k}\cdot\vec{Q})^2}{|\vec{k}|^2 (\omega - \vec{k}\cdot\vec{Q})} = 0 \tag{118}$$

This gives the $(\omega, \vec{k})$-relation for the longitudinal modes. The creation operator is then

$$\alpha_L(\vec{k}) = \Pi(\vec{k}) - \frac{i}{|\vec{k}|^2} \int d\Omega \ \left[\frac{\vec{k}\cdot\vec{Q}}{(\omega + \vec{k}\cdot\vec{Q})} J_+(\vec{k}) - \frac{\vec{k}\cdot\vec{Q}}{(\omega - \vec{k}\cdot\vec{Q})} J_-(\vec{k}) \right] \tag{119}$$

A similar analysis can be done for the transverse modes, the creation operator being

$$\alpha_T(\vec{k}) = \Pi_\lambda(\vec{k}) + i \frac{|\vec{k}|^2}{\omega} a_\lambda(\vec{k}) - i \int d\Omega \ \left[\frac{\vec{Q}\cdot e^{(\lambda)}}{(\omega + \vec{k}\cdot\vec{Q})} J_+(\vec{k}) - \frac{\vec{Q}\cdot e^{(\lambda)}}{(\omega - \vec{k}\cdot\vec{Q})} J_-(\vec{k}) \right] \tag{120}$$

So far we have considered plasma waves in Abelian subgroups. One can also construct more general solutions [29]. Consider fields of the form $A^a_\mu = A^a_\mu(p\cdot x)$, i.e., the field depends only on the combination $s = p\cdot x$ where p^μ is a timelike vector. The zero-curvature condition (75) can be solved as

$$a_+ = \tfrac{1}{2} \dot{A} \cdot Q' \ \frac{p\cdot Q}{p\cdot Q'} \tag{121}$$

where $\dot{A} = \frac{dA}{ds}$. Writing $A_\mu = \epsilon_\mu(p) h_a(s)(-it^a)$, we find the equations of motion,

$$p^\mu p_\mu \ddot{h}_3 \ + \Omega_L^2 h_3 = 0 \tag{122a}$$

$$p^\mu p_\mu \ddot{h}_1 \ + \Omega_T^2 h_1 + (h_2^2 + h_3^2) h_1 = 0 \tag{122b}$$

(We have taken the gauge group to be $SU(2)$ for simplicity; there are also equations with cyclic permutations of the labels.) Define

$$k^\mu = \frac{p^\mu \Omega}{\sqrt{p^\alpha p_\alpha}}, \qquad\qquad k^2 = \Omega^2, \tag{123}$$

for transverse and longitudinal cases. The longitudinal mode has a solution of the form $e^{ik\cdot x}$; the relation between k^0 and $\vec{k}$ is given by $k^\mu k_\mu = \Omega_L^2$, which is the same relation as for the Abelian waves. For the transverse modes, the $(k^0, \vec{k})$-relation is again the same as for the Abelian case; the amplitudes depend on $\tau = k\cdot x$ and obey the nonlinear equations

$$\ddot{f}_a + [1 + (\epsilon_{ab} f_b)^2] f_a = 0, \qquad a, b = 1, 2 \tag{124}$$

where now $\dot{f} = \frac{df}{d\tau}$. These have been further studied in ref.[29].

11. Magnetic screening and magnetic mass

Hard thermal loops and the effective action are important in setting up the resummed perturbation theory. Now, $\Gamma[A]$ is also an electric mass term for gluons and properly incorporating $\Gamma[A]$ in any calculation eliminates some of the infrared singularities. However, there would still remain some singularities since the static magnetic interactions are not screened. It is possible to introduce an infrared cutoff by hand to screen the magnetic interactions and make perturbative calculations well defined. Interestingly, a gauge invariant magnetic screening term can also be constructed using Chern-Simons related techniques [30]. Of course, in principle, one should not have to introduce such a cutoff by hand. It is generally believed that for QCD at high temperatures there is a magnetic mass term which screens the static magnetic interactions (or more generally magnetic fields with spacelike momenta). One way to understand how this might happen is as follows [31]. In the standard imaginary-time formalism for equilibrium statistical mechanics, bosonic fields are periodic in the imaginary time τ with period $1/T$, i.e., $\phi(\vec{x}, \tau + 1/T) = \phi(\vec{x}, \tau)$. The propagator has the form

$$G(x, y) = T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip(x-y)}}{(\omega_n^2 + |\vec{p}|^2)} \tag{125}$$

$\omega_n = 2\pi n T,\ n = 0, \pm 1, \pm 2, ...$, are the Matsubara frequencies. At high temperatures, because of the structure of the denominator, we expect only $T = 0$ mode to be important. There is no τ-dependence for this mode, and thus we expect QCD to reduce to three-dimensional QCD. The coupling constant of this reduced theory is $\sqrt{g^2 T}$. For QCD in three dimensions we expect a mass gap ($\sim g^2 T$) and this is effectively the magnetic mass of the high temperature four-dimensional QCD. The magnetic mass term is expected to be gauge invariant and parity even, although not necessarily local. It must be Lorentz invariant if we include the overall motion of the plasma. Also we expect such a term to be relevant only for the spatial components of the gauge potential. A term which has all these properties is given by

$$\tilde{\Gamma}[A] = -M^2 \int d\Omega\ K[A_n, A_{\bar{n}}] \tag{126}$$

where $A_n = \frac{1}{2} A_i n_i,\ A_{\bar{n}} = \frac{1}{2} A_i \bar{n}_i$ and

$$n_i = (-\cos\theta\cos\varphi - i\sin\varphi,\ -\cos\theta\sin\varphi + i\cos\varphi,\ \sin\theta\ ) \tag{127}$$

(This is in the rest frame of the plasma.) Notice that n_i is a complex three-dimensional null vector; it takes over the role of the null vector Q^μ of the electric mass term. $\tilde{\Gamma}$ involves, in the rest frame, only the spatial components of A_μ as expected for a magnetic mass term. $I(A_n)$, $I(A_{\bar{n}})$ are defined using $n \cdot x$ and $\bar{n} \cdot x$ in place of $z, \bar{z}$ in Eq.(64). Thus K in Eq.(126) is $K[\frac{1}{2}A \cdot Q, \frac{1}{2}A \cdot Q']$ of Eq.(68) with $Q^\mu \to (0, n_i)$, $Q'^\mu \to (0, \bar{n}_i)$.

Consider the simplification of the quadratic term in Eq.(126). Using

$$
\begin{aligned}
\int d\Omega \; n_i \bar{n}_j &= \frac{8\pi}{3} \delta_{ij} \\
\int d\Omega \; \frac{k \cdot \bar{n}}{k \cdot n} n_i n_j &= \frac{8\pi}{3} \left[\frac{k_i k_j}{\vec{k}^2} - \frac{1}{2} \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \right) \right]
\end{aligned}
\tag{128}
$$

we find

$$
\tilde{\Gamma} = -\frac{M^2}{2} \int \frac{d^4k}{(2\pi)^4} \; A_i^a(-k) \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \right) A_j^a(k) \; + \mathcal{O}(A^3) \tag{129}
$$

Thus $\tilde{\Gamma}$ does give screening of transverse magnetic interactions, with a screening mass M.

The higher order terms in $\tilde{\Gamma}$ can also be evaluated in a straightforward fashion, noting that the basic change is replacing Q^μ by $(0, n_i)$ and Q'^μ by $(0, \bar{n}_i)$.

Of course, there can be other ways of representing magnetic screening. What is notable about the above way of writing the magnetic screening effects is its evident kinship to the electric mass term and the Chern-Simons eikonal. Actually, in an arbitrary frame, both Γ and $\tilde{\Gamma}$ look the same, except for the choice of different orbits of the Lorentz group. Of course, we do not know what M is; no calculation even with summations of an infinity of diagrams exists at this point. There are some estimates based on lattice and other approaches to three-dimensional QCD. Even without a value for M, one can use $\tilde{\Gamma}$ as a gauge-invariant infrared cutoff in perturbative caluclations. One can also try to estimate M by some self-consistent method. Recall that for the Nambu-Jona Lasinio model [32], once we know the spinor structure of the mass term, viz., $m\bar{q}q$, we can write

$$
\mathcal{L} = \mathcal{L}_{NJL} + m\bar{q}q - m\bar{q}q \equiv \mathcal{L}_0 - m\bar{q}q \tag{130}
$$

We then calculate the one-loop effective action using $\mathcal{L}_0$, taking the subtracted term $-m\bar{q}q$ formally as a one-loop term. The vanishing of the correction to the mass then becomes the gap equation determining m. A similar calculation can be attempted here. The trouble is that one cannot really stop with the one-loop term; contributions from all loops are generally significant. It is possible that the self-consistency equations will simplify in some cases, such as perhaps the planar diagram approximation.

12. Acknowledgements

Finally I thank Professor J.E. Kim and others involved in organizing the Mt. Sorak summer school for inviting me to give these lectures. The summer school

was an occasion, both pleasant and intellectually stimulating, and it gave me the opportunity to get to know a great country and a nice people.

This work was supported in part by NSF grant PHY-9322591.

13. References

1. R.Efraty and V.P.Nair, *Phys.Rev.Lett.* **68** (1992) 2891; *Phys.Rev.* **D47** (1993) 5601.
2. R.Jackiw and V.P.Nair, *Phys.Rev.* **D48** (1993) 4991.
3. V.P.Nair, *Phys.Rev.* **D48** (1993) 3432; *ibid.* **D50** (1994) 4201.
4. B.Svetitsky and L.Yaffe, *Nucl.Phys.* **B210** (1982) 423 and references therein; R.Pisarski and F.Wilczek, *Phys.Rev.* **D29** (1984) 338.
5. For a recent review, see B.Petersson, *Nucl.Phys. (Proc.Suppl.)* **B30** (1993) 66.
6. See, for example, C.P.Singh, *Phys.Rep.* **236** (1993) 147.
7. Reference 6; K.Rajagopal and F.Wilczek, *Nucl.Phys.* **B379** (1993) 395; *ibid.* **B404** (1993) 577; articles by R.Pisarski and U.Heinz in *Banff/CAP Workshop on Thermal Field Theory*, F.C.Khanna, *et al* (eds.) (World Scientific, Singapore, 1994).
8. V.P.Silin, *Sov.J.Phys.JETP* **11** (1960) 1136; V.N.Tsytovich, *Sov.Phys.JETP*, **13** (1961) 1249; O.K.Kalashnikov and V.V.Klimov, *Sov.J.Nucl.Phys.*, **31** (1980) 699; V.V.Klimov, *Sov.J.Nucl.Phys.* **33** (1981) 934; *Sov.Phys.JETP* **55** (1982) 199; H.A.Weldon, *Phys.Rev.D* **26** (1982) 1394.
9. R.Pisarski, *Physica A* **158** (1989) 246; *Phys.Rev.Lett.* **63** (1989) 1129; E.Braaten and R.Pisarski, *Phys.Rev.* **D42** (1990) 2156; *Nucl.Phys.* **B337** (1990) 569; *ibid.* **B339** (1990) 310; *Phys.Rev.* **D45** (1992) 1827.
10. R.Kobes, G.Kunstatter and A.Rebhan, *Nucl.Phys.* **B355** (1991) 1.
11. C.Itzykson and J.B.Zuber, *Quantum Field Theory* (McGraw Hill, New York, 1980) pp.594-599.
12. J.Frenkel and J.C.Taylor, *Nucl.Phys.* **B334** (1990) 199; J.C.Taylor and S.M.H.Wong, *Nucl.Phys.* **B346** (1990) 115.
13. R.Jackiw and S.Templeton, *Phys.Rev.* **D23** (1981) 2291; J.Schonfeld, *Nucl. Phys.* **B185** (1981) 157; S.Deser, R.Jackiw and S.Templeton, *Phys.Rev.Lett.* **48** (1982) 975; *Ann.Phys.* **140** (1982) 372.
14. E.Witten, *Commun.Math.Phys.* **121** (1989) 351.
15. D.Gonzales and A.N.Redlich, *Ann.Phys.(N.Y.)* **169** (1986) 104; B.M.Zupnik, *Phys.Lett.* **B183** (1987) 175; G.V.Dunne, R.Jackiw and C.A.Trugenberger, *Ann.Phys.(N.Y.)* **149** (1989) 197; M.Bos and V.P.Nair, *Phys.Lett.* **B223** (1989) 61; *Int.J.Mod.Phys.* **A5** (1990) 959; S.Elitzur, G.Moore, A.Schwimmer and N.Seiberg, *Nucl.Phys.* **B326** (1989) 108; A.P. Polychronakos, *Ann. Phys.* **203** (1990) 231.
16. S.P. Novikov, *Usp.Mat.Nauk* **37** (1982) 3; E. Witten, *Commun.Math.Phys.* **92** (1984), 455; V.G.Knizhnik and A.B.Zamolodchikov, *Nucl.Phys.* **B247**

(1984) 83.
17. A.M.Polyakov and P.B.Wiegmann, *Phys.Lett.* **141B** (1984) 223.
18. See, for example, S.S.Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1964) pp-742-764.
19. J.Schwinger, *J.Math.Phys.* **2** (1961) 407; P.M.Bakshi and K.Mahanthappa, *J.Math.Phys.* **4** (1963) 1; *ibid.* **4** (1963) 12; L.V.Keldysh, *Sov.Phys. JETP* **20** (1964) 1018.
20. J.P.Blaizot and E.Iancu, *Phys.Rev.Lett.* **70** (1993) 3376; *Nucl.Phys.* **B 417** (1994) 608.
21. R.Jackiw, Q.Liu and C.Lucchesi, *Phys.Rev.* **D49** (1994) 6787.
22. P.F.Kelly *et al*, *Phys.Rev.Lett.* **72** (1994) 3461; *Phys.Rev.* **D50** (1994) 4209.
23. S.Wong, *Nuovo Cim.* **65A** (1970) 689.
24. See, for example, A.W.Knapp, *Lie Groups, Lie Algebras and Cohomology* (Princeton University Press, Princeton, 1988); A.Pressley and G.Segal, *Loop Groups* (Oxford: Clarendon Press, 1986); J.Sniatycki, *Geometric Quantization and Quantum Mechanics* (Springer-Verlag, New York, 1980).
25. U.Heinz, *Phys.Rev.Lett.* **51** (1983) 351; *Ann.Phys. (N.Y.)* **161** (1985) 48; *Ann.Phys. (N.Y.)* **168** (1986) 148; H.-Th.Elze and U.Heinz, *Phys.Rep.* **183** (1989) 81.
26. E.Braaten, in *Hot Summer Daze*, A.Gocksch and R.Pisarski (eds.) (World Scientific, Singapore, 1992).
27. R.I.Nepomechie, *Phys.Rev.* **D33** (1986) 3670; D.Karabali, Q-H.Park, H.J.Schnitzer and Z.Yang, *Phys.Lett.* **216B** (1989) 307; D.Karabali and H.J.Schnitzer, *Nucl.Phys.* **B329** (1990) 649; K.Gawedzki and A.Kupianen, *Phys.Lett.* **215B** (1988) 119; *Nucl.Phys.* **B320** (1989) 649.
28. R.Pisarski, *Physica* **A158** (1989) 146; T.Appelquist and R.Pisarski, *Phys. Rev* **D23** (1981) 2305.
29. J.P.Blaizot and E.Iancu, *Phys.Lett.* **B326** (1994) 138.
30. V.P.Nair, Preprint CCNY-HEP 94/4, May 1994.
31. A.D. Linde, *Phys.Lett.* **B96** (1980) 289; D. Gross, R. Pisarski and L. Yaffe, *Rev.Mod.Phys.* **53** (1981) 43.
32. Y.Nambu and G.Jona-Lasinio, *Phys.Rev.* **122** (1961) 345; *ibid.* **124** (1961) 246.